Discrete multitime multiple recurrence

Cristian Ghiu[1], Raluca Tuligă[2], Constantin Udrişte[2]

[1]University Politehnica of Bucharest, Faculty of Applied Sciences,
Department of Mathematical Methods and Models, Splaiul Independentei 313,
Bucharest 060042, Romania; e-mail: crisghiu@yahoo.com
[2]University Politehnica of Bucharest, Faculty of Applied Sciences, Department of Mathematics-Informatics, Splaiul Independentei 313, Bucharest 060042, Romania;
e-mails: ralucacoada@yahoo.com; udriste@mathem.pub.ro

Abstract

The aim of our paper is to formulate and solve problems concerning multitime multiple recurrence equations. We discuss in detail the generic properties and the existence and uniqueness of solutions. Among the general things, we discuss in detail the cases of autonomous and non-autonomous recurrences, highlighting in particular the theorems of existence and uniqueness of solutions. Finally, are given interesting examples which are the analogue of arithmetic progression and the analogue of geometric progression. The multitime multiple recurrences are required in analysis of algorithms, computational biology, information theory, queueing theory, filters theory, statistical physics etc. The theoretical part about them is little or not known, this being the first paper about the subject.

AMS Subject Classification (2010): 65Q99.
Keywords: multitime multiple recurrence, autonomous multitime recurrence, non-autonomous multitime recurrence.

1 General statements

A multivariate recurrence relation is an equation that recursively defines a multivariate sequence, once one or more initial terms are given: each further term of the sequence is defined as a function of the preceding terms. Some simply defined recurrence relations can have very complex (chaotic) behaviors, and they are a part of the field of mathematics known as nonlinear

analysis. We can use such recurrences including the Differential Transform Method to solve completely integrable first order PDEs system with initial conditions via discretization.

In this paper we shall refer to *discrete multitime multiple recurrence* (autonomous and non-autonomous), giving original results regarding generic properties and existence and uniqueness of solutions. Also, we seek to provide a fairly thorough and unified exposition of efficient recurrence relations in both univariate and multivariate settings. The scientific sources used by us are: filters theory [1], [3], [5]-[6], [10]-[14], general recurrence theory [9], [2], [4], [20], our results regarding the diagonal multitime recurrence [7]-[8], and multitime dynamical systems [15]-[19].

Let $m \geq 1$ be an integer number. We denote $\mathbf{1} = (1, 1, \dots, 1) \in \mathbb{Z}^m$. Also, for each $\alpha \in \{1, 2, \dots, m\}$, we denote $1_\alpha = (0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{Z}^m$, i.e., 1_α has 1 on the position α and 0 otherwise.

On $\mathbb{Z}^m$, we define the relation " $\leq$ ": for $t = (t^1, \dots, t^m)$, $s = (s^1, \dots, s^m)$,

$$s \leq t \ \text{ if } \ s^\alpha \leq t^\alpha, \ \forall \alpha \in \{1, 2, \dots, m\}.$$

One observes that " $\leq$ " is a partial order relation on $\mathbb{Z}^m$.

Let M be an arbitrary nonvoid set and $t_1 \in \mathbb{Z}^m$ be a fixed element. We consider the functions $F_\alpha \colon \left\{t \in \mathbb{Z}^m \,\middle|\, t \geq t_1\right\} \times M \to M$, $\alpha \in \{1, 2, \dots, m\}$.

We fix $t_0 \in \mathbb{Z}^m$, $t_0 \geq t_1$. A first order multitime recurrence of the type

$$x(t + 1_\alpha) = F_\alpha(t, x(t)), \quad \forall t \in \mathbb{Z}^m, \ t \geq t_0, \ \forall \alpha \in \{1, 2, \dots, m\}, \tag{1}$$

is called a *discrete multitime multiple recurrence.*

This model of multiple recurrence can be justified by the fact that a completely integrable first order PDE system

$$\frac{\partial x^i}{\partial t^\alpha}(t) = X^i_\alpha(t, x(t)), \, t \in \mathbb{R}^m$$

can be discretized as

$$x^i(t + 1_\alpha) = F^i_\alpha(t, x(t)), \, t \in \mathbb{Z}^m.$$

The initial (Cauchy) condition, for the PDE system, is translated into initial condition for the multiple recurrence.

Proposition 1. *If for any* $(t_0, x_0) \in \left\{t \in \mathbb{Z}^m \,\middle|\, t \geq t_1\right\} \times M$*, there exists at least one solution* $x \colon \left\{t \in \mathbb{Z}^m \,\middle|\, t \geq t_0\right\} \to M$ *which verifies the recurrence* (1) *and the initial condition* $x(t_0) = x_0$*, then*

$$F_\alpha(t + 1_\beta, F_\beta(t, x)) = F_\beta(t + 1_\alpha, F_\alpha(t, x)), \qquad \forall t \geq t_1, \ \forall x \in M, \tag{2}$$
$$\forall \alpha, \beta \in \{1, 2, \dots, m\}.$$

Proof. Let $t \geq t_0$. The equality $x(t+1_\beta+1_\alpha) = x(t+1_\alpha+1_\beta)$ is equivalent to

$$F_\alpha(t+1_\beta, x(t+1_\beta)) = F_\beta(t+1_\alpha, x(t+1_\alpha))$$
$$\Longleftrightarrow F_\alpha\big(t+1_\beta, F_\beta(t,x(t))\big) = F_\beta\big(t+1_\alpha, F_\alpha(t,x(t))\big).$$

For $t = t_0$, one obtains

$$F_\alpha\big(t_0+1_\beta, F_\beta(t_0,x_0)\big) = F_\beta\big(t_0+1_\alpha, F_\alpha(t_0,x_0)\big).$$

Since t_0 and x_0 are arbitrary, it follows the relations (2), $\forall \alpha, \beta$. $\square$

2 Autonomous discrete multitime multiple recurrence

Let M be a nonvoid set. For any function $G\colon M \to M$, we denote

$$G^{(n)} = \begin{cases} \underbrace{G \circ G \circ \ldots \circ G}_{n}, & \text{if } n \geq 1; \\ \mathrm{Id}_M, & \text{if } n = 0. \end{cases}$$

2.1 Existence and uniqueness Theorem

Theorem 1. *We consider the functions $G_\alpha\colon M \to M$, $\alpha \in \{1,2,\ldots,m\}$.*
a) Let $t_0 \in \mathbb{Z}^m$. If for any $x_0 \in M$, there exists at least one function

$$x\colon \{t \in \mathbb{Z}^m \mid t \geq t_0\} \to M,$$

which verifies the recurrence equation

$$x(t+1_\alpha) = G_\alpha(x(t)), \quad \forall t \geq t_0,\ \forall \alpha \in \{1,2,\ldots,m\}, \tag{3}$$

and the initial condition $x(t_0) = x_0$, then

$$G_\alpha \circ G_\beta = G_\beta \circ G_\alpha, \quad \forall \alpha, \beta \in \{1,2,\ldots,m\}. \tag{4}$$

b) If, for any $\alpha, \beta \in \{1,2,\ldots,m\}$, the relations (4) are satisfied, then, for any $(t_0, x_0) \in \mathbb{Z}^m \times M$, there exists a unique m-sequence $x\colon \{t \in \mathbb{Z}^m \mid t \geq t_0\} \to M$ which verifies the recurrence (3) and the initial condition $x(t_0) = x_0$; this sequence is definite by the formula

$$x(t) = G_1^{(t^1-t_0^1)} \circ G_2^{(t^2-t_0^2)} \circ \ldots \circ G_m^{(t^m-t_0^m)}(x_0), \quad \forall t \geq t_0. \tag{5}$$

Proof. a) The equality $x(t_0+1_\beta+1_\alpha)=x(t_0+1_\alpha+1_\beta)$ is equivalent to

$$G_\alpha(x(t_0+1_\beta))=G_\beta(x(t_0+1_\alpha))\Longleftrightarrow G_\alpha(G_\beta(x(t_0)))=G_\beta(G_\alpha(x(t_0)))$$
$$\Longleftrightarrow G_\alpha\circ G_\beta(x_0)=G_\beta\circ G_\alpha(x_0).$$

Since x_0 is arbitrary, it follows the relations (4), $\forall\alpha,\beta$.

b) Firstly we remark that any sequence of the form (5) verifies the relations (3) and the initial condition $x(t_0)=x_0$:

$$x(t+1_\alpha)=G_1^{(t^1-t_0^1)}\circ\ldots\circ G_\alpha^{(t^\alpha+1-t_0^\alpha)}\circ\ldots\circ G_m^{(t^m-t_0^m)}(x_0); \tag{6}$$

using (4) and the relation (6), it follows

$$x(t+1_\alpha)=G_\alpha\circ G_1^{(t^1-t_0^1)}\circ\ldots\circ G_\alpha^{(t^\alpha-t_0^\alpha)}\circ\ldots\circ G_m^{(t^m-t_0^m)}(x_0)=G_\alpha(x(t)).$$

The initial condition $x(t_0)=x_0$ is checked immediately.

The necessity is proved by induction after m, the components number of the point $t=(t^1,...,t^m)$.

For $m=1$, we have $t=t^1$ and $t_0=t_0^1$. If $t>t_0$, then

$$x(t)=x(t^1)=G_1(x(t^1-1))=G_1^{(2)}(x(t^1-2))=$$
$$=\ldots=G_1^{(k)}(x(t^1-k))=\ldots=G_1^{(t^1-t_0^1)}(x(t_0^1))=G_1^{(t^1-t_0^1)}(x_0).$$

For $t=t_0$, the relation $x(t)=G_1^{(t^1-t_0^1)}(x_0)$ is obvious.

Let $m\geq 2$. Suppose that the relation is true for $m-1$ and we shall prove it for m. We denote $\tilde{t}=(t^2,\ldots,t^m)$; $\tilde{t}_0=(t_0^2,\ldots,t_0^m)$.

Let $\tilde{x}(\tilde{t})=x(t_0^1,\tilde{t})=x(t_0^1,t^2,\ldots,t^m)$. If $t^1>t_0^1$, then

$$x(t)=x(t^1,\tilde{t})=G_1(x(t^1-1,\tilde{t}))=G_1^{(2)}(x(t^1-2,\tilde{t}))=$$
$$=\ldots=G_1^{(k)}(x(t^1-k,\tilde{t}))=\ldots=G_1^{(t^1-t_0^1)}(x(t_0^1,\tilde{t}))=G_1^{(t^1-t_0^1)}(\tilde{x}(\tilde{t})).$$

We have proved that if $t^1>t_0^1$, then $x(t)=G_1^{(t^1-t_0^1)}(\tilde{x}(\tilde{t}))$; the relation is verified automatically also for $t^1=t_0^1$.

For $\alpha\in\{2,\ldots,m\}$, we denote $\tilde{1}_\alpha=(0,\ldots,0,1,0,\ldots,0)\in\mathbb{Z}^{m-1}$; hence $1_\alpha=(0,\tilde{1}_\alpha)$. For $\alpha\geq 2$ and $t^1=t_0^1$, the relations (3) become $x((t_0^1,\tilde{t})+(0,\tilde{1}_\alpha))=G_\alpha(x(t_0^1,\tilde{t}))$, i.e.,

$$\tilde{x}(\tilde{t}+\tilde{1}_\alpha)=G_\alpha(\tilde{x}(\tilde{t})),\quad \forall\,\tilde{t}\geq\tilde{t}_0,\ \forall\alpha\in\{2,\ldots,m\}.$$

Obviously $\tilde{x}(\tilde{t}_0)=x(t_0^1,\tilde{t}_0)=x(t_0)=x_0$. Since $\tilde{t}$ has $m-1$ components, from the induction hypothesis it follows

$$\tilde{x}(\tilde{t})=G_2^{(t^2-t_0^2)}\circ\ldots\circ G_m^{(t^m-t_0^m)}(x_0),\quad\forall\tilde{t}\geq\tilde{t}_0.$$

Consequently, for any $t\geq t_0$, we have

$$x(t)=G_1^{(t^1-t_0^1)}(\tilde{x}(\tilde{t}))=G_1^{(t^1-t_0^1)}\circ G_2^{(t^2-t_0^2)}\circ\ldots\circ G_m^{(t^m-t_0^m)}(x_0).$$

$\square$

2.2 Extension theorems

Lemma 1. *Let $G\colon M \to M$ be an arbitrary function and $t_0 \in \mathbb{Z}^m$, $\beta \in \{1, 2, \ldots, m\}$, fixed. If for any $x_0 \in M$, there exists at least one function*

$$x\colon \left\{t \in \mathbb{Z}^m \,\middle|\, t \geq t_0 - 1_\beta\right\} \to M,$$

which verifies relation

$$x(t + 1_\beta) = G(x(t)), \quad \forall t \geq t_0 - 1_\beta, \tag{7}$$

and the condition $x(t_0) = x_0$, then G is surjective (onto).

Proof. Let $y \in M$. There exists a function $x(\cdot)$ which verifies (7) and the condition $x(t_0) = y$. For $t = t_0 - 1_\beta$, one obtains $x(t_0) = G(x(t_0 - 1_\beta))$, hence $G(x(t_0 - 1_\beta)) = y$. Because y is arbitrary, it follows that the function G is surjective. □

Proposition 2. *We consider the functions $G_\alpha\colon M \to M$, for $\alpha \in \{1, 2 \ldots, m\}$.*
a) Let $t_0 \in \mathbb{Z}^m$ and $\alpha_0 \in \{1, 2, \ldots, m\}$, fixed. If for any $x_0 \in M$, there exists at least one function $x\colon \left\{t \in \mathbb{Z}^m \,\middle|\, t \geq t_0 - 1_{\alpha_0}\right\} \to M$, which verifies

$$x(t + 1_\alpha) = G_\alpha(x(t)), \tag{8}$$

$$\forall t \geq t_0 - 1_{\alpha_0},\ \forall \alpha \in \{1, 2, \ldots, m\},$$

and the condition $x(t_0) = x_0$, then G_{α_0} is surjective and

$$G_\alpha \circ G_\beta = G_\beta \circ G_\alpha, \quad \forall \alpha, \beta \in \{1, 2, \ldots, m\}. \tag{9}$$

b) Suppose that, for any $\alpha \in \{1, 2, \ldots, m\}$, the functions G_α are surjective and that, for any $\alpha, \beta \in \{1, 2, \ldots, m\}$, the relations (9) are satisfied.

Let $(t_0, x_0) \in \mathbb{Z}^m \times M$ and $s \in \mathbb{Z}^m$, $s \leq t_0$.

If for $a \in M$, we have $G_1^{(t_0^1 - s^1)} \circ G_2^{(t_0^2 - s^2)} \circ \ldots \circ G_m^{(t_0^m - s^m)}(a) = x_0$, then the function

$$x\colon \left\{t \in \mathbb{Z}^m \,\middle|\, t \geq s\right\} \to M,$$

$$x(t) = G_1^{(t^1 - s^1)} \circ G_2^{(t^2 - s^2)} \circ \ldots \circ G_m^{(t^m - s^m)}(a), \quad \forall t \geq s, \tag{10}$$

verifies the recurrence (8), $\forall t \geq s$, $\forall \alpha \in \{1, 2, \ldots, m\}$, and $x(t_0) = x_0$.
c) Suppose that, for any $\alpha \in \{1, 2, \ldots, m\}$, the functions G_α are surjective and that, for any $\alpha, \beta \in \{1, 2, \ldots, m\}$, the relations (9) are satisfied.

Then, for any $(t_0, x_0) \in \mathbb{Z}^m \times M$, there exists at least one function $x\colon \mathbb{Z}^m \to M$ which verifies the recurrence (8), $\forall t \in \mathbb{Z}^m$, $\forall \alpha \in \{1, 2, \ldots, m\}$, and the condition $x(t_0) = x_0$.

Proof. *a*) The surjectivity of G_{α_0} follows from Lemma 1. The relations (9) are obtained from Theorem 1, *a*), considering the restriction of $x(\cdot)$ to the set $\{t \in \mathbb{Z}^m \,|\, t \geq t_0\}$.

b) We observe that the function $G_1^{(t_0^1-s^1)} \circ G_2^{(t_0^2-s^2)} \circ \ldots \circ G_m^{(t_0^m-s^m)}$ is surjective, since $t_0^\alpha - s^\alpha \geq 0$, $\forall \alpha$, and G_α are surjective. Consequently, there exists $a \in M$ such that $G_1^{(t_0^1-s^1)} \circ G_2^{(t_0^2-s^2)} \circ \ldots \circ G_m^{(t_0^m-s^m)}(a) = x_0$.

From the Theorem 1, *b*), it follows the function defined by the formula (10) is the unique function which verifies the recurrence (8), $\forall t \geq s$, $\forall \alpha$, and the condition $x(s) = a$. For $t = t_0$, we have

$$x(t_0) = G_1^{(t_0^1-s^1)} \circ G_2^{(t_0^2-s^2)} \circ \ldots \circ G_m^{(t_0^m-s^m)}(a) = x_0.$$

c) Let $G = G_1 \circ G_2 \circ \ldots \circ G_m$. Since the functions G_α are surjective, it follows that the function G is surjective. Hence, there exists a function $H \colon M \to M$ such that $G \circ H = \mathrm{Id}_M$ (right inverse).

For $n \in \mathbb{N}$, we denote $P_n = \{t \in \mathbb{Z}^m \,|\, t \geq t_0 - n \cdot \mathbf{1}\}$; let $a_n = H^{(n)}(x_0)$. We observe that $G(a_{n+1}) = a_n$ and $G^{(n)}(a_n) = x_0$, $\forall n \in \mathbb{N}$.

For each $n \in \mathbb{N}$, we consider the function $y_n \colon P_n \to M$, defined by

$$y_n(t) = G_1^{(t^1-t_0^1+n)} \circ G_2^{(t^2-t_0^2+n)} \circ \ldots \circ G_m^{(t^m-t_0^m+n)}(a_n), \quad \forall t \geq t_0 - n \cdot \mathbf{1}.$$

Because $G^{(n)}(a_n) = x_0$, i.e., $G_1^{(n)} \circ G_2^{(n)} \circ \ldots \circ G_m^{(n)}(a_n) = x_0$, according the step *b*), it follows that the function y_n verifies the recurrence (8), $\forall t \in P_n$, $\forall \alpha$ and the condition $y_n(t_0) = x_0$.

We remark that $P_n \subseteq P_{n+1}$. For $t \in P_n$, we have

$$\begin{aligned} y_{n+1}(t) &= G_1^{(t^1-t_0^1+n+1)} \circ G_2^{(t^2-t_0^2+n+1)} \circ \ldots \circ G_m^{(t^m-t_0^m+n+1)}(a_{n+1}) \\ &= G_1^{(t^1-t_0^1+n)} \circ G_2^{(t^2-t_0^2+n)} \circ \ldots \circ G_m^{(t^m-t_0^m+n)}\big(G(a_{n+1})\big) \\ &= G_1^{(t^1-t_0^1+n)} \circ G_2^{(t^2-t_0^2+n)} \circ \ldots \circ G_m^{(t^m-t_0^m+n)}(a_n) = y_n(t). \end{aligned}$$

We showed that $y_{n+1}(t) = y_n(t)$, $\forall t \in P_n$. Inductively, one deduces that, for any $q \in \mathbb{N}$, we have $y_{n+q}(t) = y_n(t)$, $\forall t \in P_n$. Consequently, $y_n(t) = y_k(t)$, $\forall t \in P_{\min\{n,k\}}$.

Let us define the function $x \colon \mathbb{Z}^m \to M$.

Let $t \in \mathbb{Z}^m$. Since $\mathbb{Z}^m = \bigcup\limits_{n \in \mathbb{N}} P_n$, there exists $n \in \mathbb{N}$, such that $t \in P_n$. The value of the function x at t will be $x(t) = y_n(t)$.

The function $x(\cdot)$ is well defined since if $t \in P_n$ and $t \in P_k$, we have showed that $y_n(t) = y_k(t)$.

If $t \in P_n$, then $t + 1_\alpha \in P_n$. We have $x(t+1_\alpha) = y_n(t+1_\alpha) = G_\alpha(y_n(t)) = G_\alpha(x(t))$ and $x(t_0) = y_n(t_0) = x_0$. $\square$

Proposition 3. *Suppose that, for the functions $G_\alpha\colon M \to M$, the relations* (9) *are satisfied.*

Let $t_0 \in \mathbb{Z}^m$ and $\alpha_0 \in \{1, 2, \ldots, m\}$, fixed. If, for any $x_0 \in M$, there exists at most one function $x\colon \left\{t \in \mathbb{Z}^m \,\middle|\, t \geq t_0 - 1_{\alpha_0}\right\} \to M$, which verifies

$$x(t + 1_\alpha) = G_\alpha(x(t)), \quad \forall t \geq t_0 - 1_{\alpha_0},\ \forall \alpha \in \{1, 2, \ldots, m\}, \tag{11}$$

and the condition $x(t_0) = x_0$, then G_{α_0} is injective (one-to-one).

Proof. Let $p, q \in M$ such that $G_{\alpha_0}(p) = G_{\alpha_0}(q)$.

We select $x_0 = G_{\alpha_0}(p) = G_{\alpha_0}(q)$.

The functions

$$x, y\colon \left\{t \in \mathbb{Z}^m \,\middle|\, t \geq t_0 - 1_{\alpha_0}\right\} \to M,$$

$$x(t) = G_1^{(t^1 - t_0^1)} \circ \ldots \circ G_{\alpha_0}^{(t^{\alpha_0} - t_0^{\alpha_0} + 1)} \circ \ldots \circ G_m^{(t^m - t_0^m)}(p), \quad \forall t \geq t_0 - 1_{\alpha_0}, \tag{12}$$

$$y(t) = G_1^{(t^1 - t_0^1)} \circ \ldots \circ G_{\alpha_0}^{(t^{\alpha_0} - t_0^{\alpha_0} + 1)} \circ \ldots \circ G_m^{(t^m - t_0^m)}(q), \quad \forall t \geq t_0 - 1_{\alpha_0}, \tag{13}$$

are well defined (since $t^{\alpha_0} - t_0^{\alpha_0} + 1 \geq 0$), verifies the relations (11) and $x(t_0) = G_{\alpha_0}(p) = x_0$, $y(t_0) = G_{\alpha_0}(q) = x_0$. It follows that $x(t) = y(t)$, $\forall t \geq t_0 - 1_{\alpha_0}$. For $t = t_0 - 1_{\alpha_0}$, we obtain $x(t_0 - 1_{\alpha_0}) = y(t_0 - 1_{\alpha_0})$, relation which is equivalent to $p = q$ (according (12), (13)). Hence, the function G_{α_0} is injective. $\square$

If $G\colon M \to M$ is a bijective function, we denote $G^{(-k)} = \left(G^{-1}\right)^{(k)}$, for $k \in \mathbb{N}$; we have $G^{(-k)} = \left(G^{(k)}\right)^{-1}$.

Proposition 4. *Suppose that the functions $G_\alpha\colon M \to M$ are bijective and the relations* (9) *hold. Then, for any $(t_0, x_0) \in \mathbb{Z}^m \times M$, there exists a unique solution $x\colon \mathbb{Z}^m \to M$, of the recurrence equation*

$$x(t + 1_\alpha) = G_\alpha(x(t)), \quad \forall t \in \mathbb{Z}^m,\ \forall \alpha \in \{1, 2, \ldots, m\}, \tag{14}$$

with the condition $x(t_0) = x_0$. The function x is defined by the relation

$$x(t) = G_1^{(t^1 - t_0^1)} \circ G_2^{(t^2 - t_0^2)} \circ \ldots \circ G_m^{(t^m - t_0^m)}(x_0) \qquad (\forall t \in \mathbb{Z}^m). \tag{15}$$

Proof. The existence follows from the Proposition 2, *c*).

Let $x\colon \mathbb{Z}^m \to M$ be a solution of the recurrence (14), with $x(t_0) = x_0$. For proving the uniqueness, it is sufficient to show that $x(t)$ verifies the relation (15), $\forall t \in \mathbb{Z}^m$.

Let $s \leq t_0$. We apply the Theorem 1 for the restriction of x to the set $\{t \in \mathbb{Z}^m \,|\, t \geq s\}$. It follows

$$x(t) = G_1^{(t^1-s^1)} \circ G_2^{(t^2-s^2)} \circ \ldots \circ G_m^{(t^m-s^m)}(x(s)), \quad \forall t \geq s.$$

For $t = t_0$, we obtain $x_0 = G_1^{(t_0^1-s^1)} \circ G_2^{(t_0^2-s^2)} \circ \ldots \circ G_m^{(t_0^m-s^m)}(x(s))$. Since the functions G_α are bijective, it follows

$$x(s) = G_1^{(s^1-t_0^1)} \circ G_2^{(s^2-t_0^2)} \circ \ldots \circ G_m^{(s^m-t_0^m)}(x_0).$$

Consequently, for any $t \geq s$, we have

$$x(t) = G_1^{(t^1-s^1)} \circ G_2^{(t^2-s^2)} \circ \ldots \circ G_m^{(t^m-s^m)}(x(s)) =$$

$$= G_1^{(t^1-s^1)} \circ G_2^{(t^2-s^2)} \circ \ldots \circ G_m^{(t^m-s^m)} \circ G_1^{(s^1-t_0^1)} \circ G_2^{(s^2-t_0^2)} \circ \ldots \circ G_m^{(s^m-t_0^m)}(x_0)$$
$$= G_1^{(t^1-t_0^1)} \circ G_2^{(t^2-t_0^2)} \circ \ldots \circ G_m^{(t^m-t_0^m)}(x_0).$$

We have showed that, for any $s \leq t_0$ and any $t \geq s$, the sequence $x(t)$ verifies the relation (15). Since

$$\bigcup_{s \in \mathbb{Z}^m,\, s \leq t_0} \{t \in \mathbb{Z}^m \,|\, t \geq s\} = \mathbb{Z}^m,$$

it follows that the relation (15) holds for any $t \in \mathbb{Z}^m$. □

Theorem 2. *Let M be a nonvoid set. For each $\alpha \in \{1, 2, \ldots, m\}$, we consider the function $G_\alpha \colon M \to M$ and we associate the recurrence equation*

$$x(t + 1_\alpha) = G_\alpha(x(t)), \quad \forall \alpha \in \{1, 2, \ldots, m\}. \tag{16}$$

The following statements are equivalent:
i) For any $\alpha \in \{1, 2, \ldots, m\}$, the functions G_α are bijective and

$$G_\alpha \circ G_\beta = G_\beta \circ G_\alpha, \quad \forall \alpha, \beta \in \{1, 2, \ldots, m\}. \tag{17}$$

ii) There exists $t_0 \in \mathbb{Z}^m$ such that $\forall \alpha_0 \in \{1, 2, \ldots, m\}$, $\forall x_0 \in M$, there exists a unique function $x \colon \{t \in \mathbb{Z}^m \,|\, t \geq t_0 - 1_{\alpha_0}\} \to M$, which, for any $t \geq t_0 - 1_{\alpha_0}$, verifies the relations (16), and the condition $x(t_0) = x_0$.
iii) There exist $t_0, t_1 \in \mathbb{Z}^m$, with $t_1^\alpha < t_0^\alpha$, $\forall \alpha$, such that, for each $x_0 \in M$, there exists a unique function $x \colon \{t \in \mathbb{Z}^m \,|\, t \geq t_1\} \to M$, which, for each $t \geq t_1$ verifies the relations (16), and also the condition $x(t_0) = x_0$.

iv) For each $t_0, t_1 \in \mathbb{Z}^m$*, with* $t_1 \leq t_0$*, and for any* $x_0 \in M$*, there exists a unique function* $x\colon \{t \in \mathbb{Z}^m \,|\, t \geq t_1\} \to M$*, which, for any* $t \geq t_1$ *verifies the relations* (16)*, and also the condition* $x(t_0) = x_0$*.*

v) There exists $t_0 \in \mathbb{Z}^m$*, such that, for any* $x_0 \in M$*, there exist a unique function* $x\colon \mathbb{Z}^m \to M$*, which, for any* $t \in \mathbb{Z}^m$ *verifies the relations* (16)*, and* $x(t_0) = x_0$*.*

vi) For each pair $(t_0, x_0) \in \mathbb{Z}^m \times M$*, there exists a unique function* $x\colon \mathbb{Z}^m \to M$*, which, for any* $t \in \mathbb{Z}^m$ *verifies the relations* (16)*, and* $x(t_0) = x_0$*.*

Proof. *ii)* $\Longrightarrow$ *i)*: The relations (17) and the surjectivity of functions G_α follow from the Proposition 2, *a*), and the injectivity of the functions G_α follow from the Proposition 3.

i) $\Longrightarrow$ *vi)*: It follows from the Proposition 4.

vi) $\Longrightarrow$ *iv)*: Considering the restrictions of the functions x to $\{t \in \mathbb{Z}^m \,|\, t \geq t_0 - 1_{\alpha_0}\}$ (for each α_0), from the Proposition 2, *a*), it follows that the relations (17) hold and that the functions G_α are surjective, $\forall \alpha$.

Let $t_0, t_1 \in \mathbb{Z}^m$, with $t_1 \leq t_0$, and $x_0 \in M$. There exists a unique function $\tilde{x}\colon \mathbb{Z}^m \to M$ such that $\tilde{x}(t_0) = x_0$ and the relations (16) are true, $\forall t \in \mathbb{Z}^m$.

To prove the existence, it is sufficient to select x as the restriction of $\tilde{x}$ to $\{t \in \mathbb{Z}^m \,|\, t \geq t_1\}$.

Let $y\colon \{t \in \mathbb{Z}^m \,|\, t \geq t_1\} \to M$, be a function such that $y(t_0) = x_0$ and for which the relations (16) hold, $\forall t \geq t_1$. We shall prove that the functions x and y are equal.

From the Proposition 2, *c*), there exists $\tilde{y}\colon \mathbb{Z}^m \to M$ such that $\tilde{y}(t_1) = y(t_1)$ and for which the relations (16) hold, $\forall t \in \mathbb{Z}^m$. From the Theorem 1, it follows that y and its restriction $\tilde{y}$ to $\{t \in \mathbb{Z}^m \,|\, t \geq t_1\}$ coincide. Since $t_0 \geq t_1$, we have $\tilde{y}(t_0) = y(t_0) = x_0$. It follows that the functions $\tilde{x}$ and $\tilde{y}$ coincide. Consequently, for each $t \geq t_1$, we have

$$y(t) = \tilde{y}(t) = \tilde{x}(t) = x(t).$$

iv) $\Longrightarrow$ *ii)* is an obvious implication.

We have proved that the statements *i)*, *ii)*, *iv)*, *vi)* are equivalent.

i) $\Longrightarrow$ *iii)*: We have *i)* $\Longleftrightarrow$ *iv)*, and *iv)* $\Longrightarrow$ *iii)* is obvious.

iii) $\Longrightarrow$ *i)*: For each α, we have $t_1^\alpha < t_0^\alpha$, i.e., $t_0^\alpha - 1 \geq t_1^\alpha$. Hence $t_0 - 1_\alpha \geq t_1$, $\forall \alpha$. Considering the restrictions of the functions x to $\{t \in \mathbb{Z}^m \,|\, t \geq t_0 - 1_{\alpha_0}\}$ (for each α_0), from the Proposition 2, *a*), it follows that the relations (17) are true and the functions G_α are surjective, $\forall \alpha$.

Let $\alpha_0 \in \{1, 2, \ldots, m\}$. We shall prove that G_{α_0} is injective.

Let $p, q \in M$ such that $G_{\alpha_0}(p) = G_{\alpha_0}(q)$. According the Proposition 2, *c*), there exist the functions $y, z\colon \mathbb{Z}^m \to M$ for which the relations (16) hold,

$\forall t \in \mathbb{Z}^m$, and $y(t_0 - 1_{\alpha_0}) = p$, $z(t_0 - 1_{\alpha_0}) = q$. Let $x_0 = G_{\alpha_0}(p) = G_{\alpha_0}(q)$.

$$y(t_0) = G_{\alpha_0}(y(t_0 - 1_{\alpha_0})) = G_{\alpha_0}(p) = x_0,$$

$$z(t_0) = G_{\alpha_0}(z(t_0 - 1_{\alpha_0})) = G_{\alpha_0}(q) = x_0.$$

Applying the uniqueness property for the restrictions of the functions y and z to the set $\{t \in \mathbb{Z}^m \,|\, t \geq t_1\}$, we obtain $y(t) = z(t)$, $\forall t \geq t_1$.

Since $t_0 - 1_{\alpha_0} \geq t_1$, it follows $y(t_0 - 1_{\alpha_0}) = z(t_0 - 1_{\alpha_0})$, i.e., $p = q$.

$i) \Longrightarrow v)$: We have $i) \Longleftrightarrow vi)$, and $vi) \Longrightarrow v)$ is obvious.

$v) \Longrightarrow i)$: For each α, we have $t_1^\alpha < t_0^\alpha$, i.e., $t_0^\alpha - 1 \geq t_1^\alpha$. Hence $t_0 - 1_\alpha \geq t_1$, $\forall \alpha$. Considering the restrictions of the functions x to $\{t \in \mathbb{Z}^m \,|\, t \geq t_0 - 1_{\alpha_0}\}$ (for each α_0), by the Proposition 2, a), it follows that the relations (17) hold and that the functions G_α are surjective, $\forall \alpha$.

Let $\alpha_0 \in \{1, 2, \ldots, m\}$. We shall prove that G_{α_0} is injective.

Let $p, q \in M$ such that $G_{\alpha_0}(p) = G_{\alpha_0}(q)$. According the Proposition 2, c), there exist the functions $y, z \colon \mathbb{Z}^m \to M$ for which the relations (16) are true, $\forall t \in \mathbb{Z}^m$, and $y(t_0 - 1_{\alpha_0}) = p$, $z(t_0 - 1_{\alpha_0}) = q$. Let $x_0 = G_{\alpha_0}(p) = G_{\alpha_0}(q)$.

$$y(t_0) = G_{\alpha_0}(y(t_0 - 1_{\alpha_0})) = G_{\alpha_0}(p) = x_0,$$

$$z(t_0) = G_{\alpha_0}(z(t_0 - 1_{\alpha_0})) = G_{\alpha_0}(q) = x_0.$$

From uniqueness, we obtain $y(t) = z(t)$, $\forall t \in \mathbb{Z}^m$; for $t = t_0 - 1_{\alpha_0}$ it follows $y(t_0 - 1_{\alpha_0}) = z(t_0 - 1_{\alpha_0})$, i.e., $p = q$. $\square$

Remark 1. *We consider the bijective functions $G_\alpha \colon M \to M$, for which the relations* (17) *holds. Let $t_0 \in \mathbb{Z}^m$ and let $\tilde{x} \colon \{t \in \mathbb{Z}^m \,|\, t \geq t_0\} \to M$ be a solution of the recurrence* (16)*. From Theorem 2, it follows that there exists a unique function $x \colon \mathbb{Z}^m \to M$, solution of the recurrence* (16)*, such that $x(t) = \tilde{x}(t)$, $\forall t \geq t_0$, i.e., x is an extension of $\tilde{x}$. According Proposition* 4*, this extension is defined by the formula* (15)*, for $x_0 = \tilde{x}(t_0)$.*

3 Non-autonomous discrete multitime multiple recurrence

3.1 Existence and uniqueness Theorem

Let $t_1 \in \mathbb{Z}^m$. Consider the functions $F_\alpha \colon \{t \in \mathbb{Z}^m \,|\, t \geq t_1\} \times M \to M$, $\alpha \in \{1, 2, \ldots, m\}$, which define the recurrence equation

$$x(t + 1_\alpha) = F_\alpha(t, x(t)), \quad \forall \alpha \in \{1, 2, \ldots, m\}. \tag{18}$$

Let $\widetilde{M} = \{s \in \mathbb{Z}^m \,|\, s \geq t_1\} \times M$ and let $G_\alpha \colon \widetilde{M} \to \widetilde{M}$,

$$G_\alpha(s, x) = \big(s + 1_\alpha, F_\alpha(s, x)\big), \quad \forall (s, x) \in \widetilde{M}.$$

The functions G_α define the recurrence

$$\big(s(t+1_\alpha), x(t+1_\alpha)\big) = \big(s(t)+1_\alpha, F_\alpha(s(t), x(t))\big), \quad \forall \alpha \in \{1, 2, \dots, m\}, \tag{19}$$

which is equivalent to

$$\begin{cases} x(t + 1_\alpha) = F_\alpha(s(t), x(t)) \\ s(t + 1_\alpha) = s(t) + 1_\alpha \end{cases}, \quad \forall \alpha \in \{1, 2, \dots, m\}. \tag{20}$$

The unknown function is $\big(s(\cdot), x(\cdot)\big)$. Denoting $y = (s, x)$, the recurrence (19) can be rewritten in the form

$$y(t + 1_\alpha) = G_\alpha(y(t)), \quad \forall \alpha \in \{1, 2, \dots, m\}, \tag{21}$$

with the unknown function $y(\cdot) = \big(s(\cdot), x(\cdot)\big)$.

Lemma 2. *a) Let $t_0, t_1, s_0 \in \mathbb{Z}^m$, with $t_0 \geq t_1$.*

Then the function $s \colon \{t \in \mathbb{Z}^m \,|\, t \geq t_1\} \to \mathbb{Z}^m$ verifies, for each $t \geq t_1$,

$$s(t + 1_\alpha) = s(t) + 1_\alpha, \quad \forall \alpha \in \{1, 2, \dots, m\}, \tag{22}$$

and the condition $s(t_0) = s_0$,

if and only if $s(t) = t - t_0 + s_0$, $\forall t \geq t_1$.

b) Let $t_0, s_0 \in \mathbb{Z}^m$. The function $s \colon \mathbb{Z}^m \to \mathbb{Z}^m$ verifies, for each $t \in \mathbb{Z}^m$, the relations (22) and the condition $s(t_0) = s_0$ if and only if $s(t) = t - t_0 + s_0$, $\forall t \in \mathbb{Z}^m$.

Proof. Let $\tilde{s} \colon \mathbb{Z}^m \to \mathbb{Z}^m$, $\tilde{s}(t) = t - t_0 + s_0$, $\forall t \in \mathbb{Z}^m$. One observes immediately that $\tilde{s}$ verifies, for any $t \in \mathbb{Z}^m$, the relations (22) and $\tilde{s}(t_0) = s_0$.

For each α, we consider the function

$$\widetilde{G}_\alpha \colon \mathbb{Z}^m \to \mathbb{Z}^m, \quad \widetilde{G}_\alpha(s) = s + 1_\alpha, \ \forall s \in \mathbb{Z}^m.$$

The relations (22) are equivalent to $s(t+1_\alpha) = \widetilde{G}_\alpha(s(t))$, $\forall \alpha \in \{1, 2, \dots, m\}$.

One observes that $\widetilde{G}_\alpha \circ \widetilde{G}_\beta(s) = \widetilde{G}_\beta \circ \widetilde{G}_\alpha(s) = s + 1_\alpha + 1_\beta$, $\forall s \in \mathbb{Z}^m$.

For any α, the function $\widetilde{G}_\alpha$ is bijective. Its inverse is $(\widetilde{G}_\alpha)^{-1}(s) = s - 1_\alpha$, $\forall s \in \mathbb{Z}^m$.

According the Theorem 2, *iv*), there exists a unique function $s \colon \{t \in \mathbb{Z}^m \,|\, t \geq t_1\} \to \mathbb{Z}^m$ which verifies the recurrence (22), $\forall t \geq t_1$, and the

condition $s(t_0) = t_0$. By uniqueness, it follows that s coincides with the restriction of the function $\tilde{s}$ to the set $\{t \in \mathbb{Z}^m \mid t \geq t_1\}$; hence $s(t) = t - t_0 + s_0$, $\forall t \geq t_1$.

According the Theorem 2, *vi*), it follows that there exists a unique function $\sigma\colon \mathbb{Z}^m \to \mathbb{Z}^m$ which verifies the recurrence (22), $\forall t \in \mathbb{Z}^m$, and the condition $\sigma(t_0) = t_0$. From uniqueness, it follows that $\sigma = \tilde{s}$; hence $\sigma(t) = t - t_0 + s_0$, $\forall t \in \mathbb{Z}^m$. $\square$

Proposition 5. *Suppose that we are in the foregoing conditions.*
a) For $\alpha, \beta \in \{1, 2, \ldots, m\}$, *we have*

$$F_\alpha(t + 1_\beta, F_\beta(t, x)) = F_\beta(t + 1_\alpha, F_\alpha(t, x)), \qquad \forall t \geq t_1,\ \forall x \in M \tag{23}$$

if and only if $G_\alpha \circ G_\beta = G_\beta \circ G_\alpha$.
b) Let $t_0 \in \mathbb{Z}^m$, $t_0 \geq t_1$ *and* $x_0 \in M$.

If $x\colon \{t \in \mathbb{Z}^m \mid t \geq t_1\} \to M$ *verifies the recurrence* (18), $\forall t \geq t_1$, *and the condition* $x(t_0) = x_0$, *then the function*

$$y\colon \{t \in \mathbb{Z}^m \mid t \geq t_1\} \to \widetilde{M}, \quad y(t) = (t, x(t)), \quad \forall t \geq t_1,$$

verifies the recurrence (21), $\forall t \geq t_1$, *and the condition* $y(t_0) = (t_0, x_0)$.

Conversely, if $y(\cdot) = \big(s(\cdot), x(\cdot)\big)\colon \{t \in \mathbb{Z}^m \mid t \geq t_1\} \to \widetilde{M}$ *verifies the recurrence* (21), $\forall t \geq t_1$, *and the condition* $y(t_0) = (t_0, x_0)$, *then* $s(t) = t$, $\forall t \geq t_1$ *and* $x(\cdot)$ *verifies the recurrence* (18), $\forall t \geq t_1$, *and the condition* $x(t_0) = x_0$.

Proof. *a*) For any $(s, x) \in \widetilde{M}$, we have

$$G_\alpha \circ G_\beta(s, x) = G_\beta \circ G_\alpha(s, x) \Longleftrightarrow$$

$$\Longleftrightarrow \big(s + 1_\beta + 1_\alpha, F_\alpha(s + 1_\beta, F_\beta(s, x))\big) = \big(s + 1_\alpha + 1_\beta, F_\beta(s + 1_\alpha, F_\alpha(s, x))\big)$$
$$\Longleftrightarrow F_\alpha(s + 1_\beta, F_\beta(s, x)) = F_\beta(s + 1_\alpha, F_\alpha(s, x))$$

b) Let $x(\cdot)$ be a solution of the recurrence (18), with $x(t_0) = x_0$. We have to show that the function $y(t) = (t, x(t))$ verifies the relations (20); since, for that $y(\cdot)$ we have $s(t) = t$, the relations (20) become

$$\begin{cases} x(t + 1_\alpha) = F_\alpha(t, x(t)) \\ t + 1_\alpha = t + 1_\alpha \end{cases}, \quad \forall \alpha \in \{1, 2, \ldots, m\}. \tag{24}$$

The second relation in (24) is obvious, and the first is true because $x(\cdot)$ is a solution of the recurrence (18).

The relation $y(t_0) = (t_0, x_0)$ is obvious.

Conversely, let $y(\cdot) = \big(s(\cdot), x(\cdot)\big)$ be a solution of the recurrence (21), with $y(t_0) = (t_0, x_0)$. Hence $s(\cdot)$ and $x(\cdot)$ verifies the relations (20) and the condition $s(t_0) = t_0$, $x(t_0) = x_0$.

Since $s(t + 1_\alpha) = s(t) + 1_\alpha$, $\forall t \geq t_1$, $\forall \alpha$, and $s(t_0) = t_0$, from Lemma 2 it follows that $s(t) = t$, $\forall t \geq t_1$.

Hence, the first relation in (20) becomes $x(t + 1_\alpha) = F_\alpha(t, x(t))$, i.e., $x(\cdot)$ is solution of the recurrence (18). $\square$

The Proposition 1 presents necessary conditions for the existence of solutions of a discrete multitime multiple recurrence. The next Theorem shows that these conditions are also sufficient for the existence and uniqueness of solutions.

Theorem 3. *Let M be an arbitrary nonvoid set and $t_0 \in \mathbb{Z}^m$. We consider the functions $F_\alpha \colon \big\{t \in \mathbb{Z}^m \,\big|\, t \geq t_0\big\} \times M \to M$, $\alpha \in \{1, 2, \ldots, m\}$, such that*

$$F_\alpha(t + 1_\beta, F_\beta(t, x)) = F_\beta(t + 1_\alpha, F_\alpha(t, x)), \tag{25}$$

$$\forall t \geq t_0,\ \forall x \in M, \quad \forall \alpha, \beta \in \{1, 2, \ldots, m\}.$$

Then, for any $x_0 \in M$, there exists a unique function $x \colon \big\{t \in \mathbb{Z}^m \,\big|\, t \geq t_0\big\} \to M$ which verifies

$$x(t + 1_\alpha) = F_\alpha(t, x(t)), \quad \forall t \geq t_0, \quad \forall \alpha \in \{1, 2, \ldots, m\}, \tag{26}$$

and the condition $x(t_0) = x_0$.

Proof. Let $\widetilde{M} = \big\{s \in \mathbb{Z}^m \,\big|\, s \geq t_0\big\} \times M$ and let $G_\alpha \colon \widetilde{M} \to \widetilde{M}$,

$$G_\alpha(s, x) = \big(s + 1_\alpha, F_\alpha(s, x)\big), \quad \forall (s, x) \in \widetilde{M}.$$

We apply the Proposition 5 (for $t_1 = t_0$); according the step a), it follows that $G_\alpha \circ G_\beta = G_\beta \circ G_\alpha$, $\forall \alpha \in \{1, 2, \ldots, m\}$. From Theorem 1, b), it follows that there exists a unique function $y(\cdot) = \big(s(\cdot), x(\cdot)\big) \colon \big\{t \in \mathbb{Z}^m \,\big|\, t \geq t_0\big\} \to \widetilde{M}$ which verifies

$$y(t + 1_\alpha) = G_\alpha(y(t)), \quad \forall t \geq t_0, \quad \forall \alpha \in \{1, 2, \ldots, m\}, \tag{27}$$

and the condition $y(t_0) = (t_0, x_0)$. From Proposition 5, b), it follows that $x(\cdot)$ verifies the relations (26) and the condition $x(t_0) = x_0$.

Uniqueness of $x(\cdot)$: let $\tilde{x} \colon \big\{t \in \mathbb{Z}^m \,\big|\, t \geq t_0\big\} \to M$ a function which verifies the relations (26) and the condition $\tilde{x}(t_0) = x_0$. From Proposition 5, b), it follows that the function

$$\tilde{y} \colon \big\{t \in \mathbb{Z}^m \,\big|\, t \geq t_0\big\} \to \widetilde{M}, \quad \tilde{y}(t) = (t, \tilde{x}(t)), \quad \forall t \geq t_0,$$

verifies the relations (27) and the condition $\tilde{y}(t_0) = (t_0, x_0)$.

From the uniqueness property of the solution of the recurrence (27) (Theorem 1, b)), it follows that the functions y and $\tilde{y}$ coincide; hence $\big(s(t), x(t)\big) = \big(t, \tilde{x}(t)\big)$, $\forall t \geq t_0$; we obtain $x(t) = \tilde{x}(t)$, $\forall t \geq t_0$. $\square$

Remark 2. *Let $t_0, t_1, s_0 \in \mathbb{Z}^m$, $s_0 \geq t_1$ and $x_0 \in M$.*

Let $\widetilde{M} = \big\{s \in \mathbb{Z}^m \,\big|\, s \geq t_1\big\} \times M$.

Suppose that $\tilde{x}\colon \big\{t \in \mathbb{Z}^m \,\big|\, t \geq s_0\big\} \to M$ verifies, for any $t \geq s_0$, the recurrence (18)*, and the condition $\tilde{x}(s_0) = x_0$. Let*

$$s, x\colon \big\{t \in \mathbb{Z}^m \,\big|\, t \geq t_0\big\} \to M,$$

$$s(t) = t - t_0 + s_0, \quad x(t) = \tilde{x}(t - t_0 + s_0), \quad \forall t \geq t_0.$$

Easily be observed that the function $y(\cdot) = \big(s(\cdot), x(\cdot)\big)\colon \big\{t \in \mathbb{Z}^m \,\big|\, t \geq t_0\big\} \to \widetilde{M}$ verifies, for each $t \geq t_0$, the recurrence (21) *and the condition $y(t_0) = (s_0, x_0)$.*

From here one obtains a new proof for the Proposition 1*; Indeed, in the hypotheses of Proposition* 1*, it follows that $\forall (s_0, x_0) \in \widetilde{M}$, there exists a function $y(\cdot)$, which verifies, for each $t \geq t_0$, the recurrence* (21) *and the condition $y(t_0) = (s_0, x_0)$. From Theorem* 1*, it follows that $G_\alpha \circ G_\beta = G_\beta \circ G_\alpha$, $\forall \alpha, \beta$, and from Proposition* 5 *we obtain the relations* (23)*.*

3.2 Extension theorems

Proposition 6. *Let $\alpha_0 \in \{1, 2, \ldots, m\}$, $t_0 \in \mathbb{Z}^m$.*
a) Let $F\colon \big\{t \in \mathbb{Z}^m \,\big|\, t \geq t_0 - 1_{\alpha_0}\big\} \times M \to M$. If, for any $x_0 \in M$, there exists at least one function $x\colon \big\{t \in \mathbb{Z}^m \,\big|\, t \geq t_0 - 1_{\alpha_0}\big\} \to M$, which verifies

$$x(t + 1_{\alpha_0}) = F(t, x(t)), \quad \forall t \geq t_0 - 1_{\alpha_0}, \tag{28}$$

and the condition $x(t_0) = x_0$, then the function $F(t_0 - 1_{\alpha_0}, \cdot)$ is surjective.
b) Suppose that for the functions $F_\alpha\colon \big\{t \in \mathbb{Z}^m \,\big|\, t \geq t_0 - 1_{\alpha_0}\big\} \times M \to M$, the relations (25) *hold, $\forall t \geq t_0 - 1_{\alpha_0}$, $\forall x \in M$, $\forall \alpha, \beta \in \{1, 2, \ldots, m\}$. If, for any $x_0 \in M$, there exists at most one function $x\colon \big\{t \in \mathbb{Z}^m \,\big|\, t \geq t_0 - 1_{\alpha_0}\big\} \to M$, which verifies*

$$x(t + 1_\alpha) = F_\alpha(t, x(t)), \quad \forall t \geq t_0 - 1_{\alpha_0},\ \forall \alpha \in \{1, 2, \ldots, m\}, \tag{29}$$

and the condition $x(t_0) = x_0$, then the function $F_{\alpha_0}(t_0 - 1_{\alpha_0}, \cdot)$ is injective.

Proof. a) Let $z \in M$. There exists a function $x(\cdot)$ which verifies (28) and the condition $x(t_0) = z$. For $t = t_0 - 1_{\alpha_0}$, one obtains $z = F(t_0 - 1_{\alpha_0}, x(t_0 - 1_{\alpha_0}))$. Since z is arbitrary, it follows that $F(t_0 - 1_{\alpha_0}, \cdot)$ is surjective.

b) Let $p, q \in M$ such that $F_{\alpha_0}(t_0 - 1_{\alpha_0}, p) = F_{\alpha_0}(t_0 - 1_{\alpha_0}, q)$. We can apply the Theorem 3. There exist the functions $x, \tilde{x} \colon \big\{ t \in \mathbb{Z}^m \,\big|\, t \geq t_0 - 1_{\alpha_0} \big\} \to M$ for which the relations (29) are true, and $x(t_0 - 1_{\alpha_0}) = p$, $\tilde{x}(t_0 - 1_{\alpha_0}) = q$.

Let $x_0 = F_{\alpha_0}(t_0 - 1_{\alpha_0}, p) = F_{\alpha_0}(t_0 - 1_{\alpha_0}, q)$. Then

$$x(t_0) = F_{\alpha_0}(t_0 - 1_{\alpha_0}, x(t_0 - 1_{\alpha_0})) = F_{\alpha_0}(t_0 - 1_{\alpha_0}, p) = x_0,$$

$$\tilde{x}(t_0) = F_{\alpha_0}(t_0 - 1_{\alpha_0}, \tilde{x}(t_0 - 1_{\alpha_0})) = F_{\alpha_0}(t_0 - 1_{\alpha_0}, q) = x_0.$$

It follows that the functions x and $\tilde{x}$ coincide; hence $x(t_0 - 1_{\alpha_0}) = \tilde{x}(t_0 - 1_{\alpha_0})$, i.e., $p = q$. $\square$

Lemma 3. *Let $\beta \in \{1, 2, \ldots, m\}$ and $F \colon \mathbb{Z}^m \times M \to M$.*

Let $G \colon \mathbb{Z}^m \times M \to \mathbb{Z}^m \times M$, $G(t, x) = \big(t + 1_\beta, F(t, x)\big)$, $\forall (t, x) \in \mathbb{Z}^m \times M$.

a) The function G is injective if and only if, for any $t \in \mathbb{Z}^m$, the function $F(t, \cdot)$ is injective.

b) The function G is surjective if and only if, for any $t \in \mathbb{Z}^m$, the function $F(t, \cdot)$ is surjective.

Proof. *a*) Suppose that the function G is injective. Let $t \in \mathbb{Z}^m$.

If, for $x_1, x_2 \in M$, we have $F(t, x_1) = F(t, x_2)$, then $G(t, x_1) = G(t, x_2)$. It follows $(t, x_1) = (t, x_2)$; hence $x_1 = x_2$.

Conversely, let us suppose that the functions $F(t, \cdot)$ are injective. If $G(t_1, x_1) = G(t_2, x_2)$, then $\big(t_1 + 1_\beta, F(t_1, x_1)\big) = \big(t_2 + 1_\beta, F(t_2, x_2)\big)$. Hence $t_1 = t_2$ and $F(t_1, x_1) = F(t_1, x_2)$. It follows that $x_1 = x_2$. We have obtained $(t_1, x_1) = (t_2, x_2)$.

b) Suppose that G is surjective. Let $t \in \mathbb{Z}^m$. If $y \in M$, then there exists $(s, x) \in \mathbb{Z}^m \times M$, such that $G(s, x) = (t + 1_\beta, y)$, equivalent to $\big(s + 1_\beta, F(s, x)\big) = \big(t + 1_\beta, y\big)$. It follows that $s = t$ and $F(s, x) = y$. Hence $F(t, x) = y$.

Conversely, let us suppose that the functions $F(t, \cdot)$ are surjective.

Let $(s, y) \in \mathbb{Z}^m \times M$. There exists $x \in M$, such that $F(s - 1_\beta, x) = y$. We have $G(s - 1_\beta, x) = \big(s, F(s - 1_\beta, x)\big) = (s, y)$. $\square$

Theorem 4. *Let M be a nonvoid set. For each $\alpha \in \{1, 2, \ldots, m\}$, we consider the function $F_\alpha \colon \mathbb{Z}^m \times M \to M$, to which we associate the recurrence equation*

$$x(t + 1_\alpha) = F_\alpha(t, x(t)), \quad \forall \alpha \in \{1, 2, \ldots, m\}. \tag{30}$$

The following statements are equivalent:

i) For any $\alpha \in \{1, 2, \ldots, m\}$ and any $t \in \mathbb{Z}^m$, the functions $F_\alpha(t, \cdot)$ are bijective and

$$F_\alpha(t + 1_\beta, F_\beta(t, x)) = F_\beta(t + 1_\alpha, F_\alpha(t, x)), \tag{31}$$

$$\forall (t,x) \in \mathbb{Z}^m \times M,\ \ \forall \alpha, \beta \in \{1, 2, \ldots, m\}.$$

ii) For any pair $(t_0, x_0) \in \mathbb{Z}^m \times M$*, and any index* $\alpha_0 \in \{1, 2, \ldots, m\}$*, there exists a unique function* $x\colon \big\{t \in \mathbb{Z}^m \,\big|\, t \geq t_0 - 1_{\alpha_0}\big\} \to M$*, which, for each* $t \geq t_0 - 1_{\alpha_0}$ *verifies the relations* (30)*, and also the condition* $x(t_0) = x_0$*.*

iii) For any $t_0, t_1 \in \mathbb{Z}^m$*, with* $t_1 \leq t_0$*, and any* $x_0 \in M$*, there exists a function* $x\colon \big\{t \in \mathbb{Z}^m \,\big|\, t \geq t_1\big\} \to M$*, which, for any* $t \geq t_1$*, verifies the relations* (30)*, and also the condition* $x(t_0) = x_0$*.*

iv) For any $(t_0, x_0) \in \mathbb{Z}^m \times M$*, there exists a unique function* $x\colon \mathbb{Z}^m \to M$*, which, for any* $t \in \mathbb{Z}^m$*, verifies the relation* (30)*, and also* $x(t_0) = x_0$*.*

Proof. *ii)* $\Longrightarrow$ *i)*: Let $t_1 \in \mathbb{Z}^m$. For any $t_0 \geq t_1$ and any $x_0 \in M$, there exists a unique function $x\colon \big\{t \in \mathbb{Z}^m \,\big|\, t \geq t_0 - 1_{\alpha_0}\big\} \to M$, which, for any $t \geq t_0 - 1_{\alpha_0}$, verifies the relations (30), and also the condition $x(t_0) = x_0$.

The restriction of the function $x(\cdot)$ to $\big\{t \in \mathbb{Z}^m \,\big|\, t \geq t_0\big\}$ verifies, for any $t \geq t_0$, the recurrence (30), and also the condition $x(t_0) = x_0$.

From the Proposition 1 it follows that the relations (31) hold, for any $t \geq t_1$. Since t_1 is arbitrary, we deduce that the relations (31) are true, for any $t \in \mathbb{Z}^m$.

The surjectivity of functions $F_\alpha(t, \cdot)$ follows from Proposition 6, *a*).

The injectivity of functions $F_\alpha(t, \cdot)$ follows from the Proposition 6, *b*).

i) $\Longrightarrow$ *iv)*: For each $\alpha \in \{1, 2, \ldots, m\}$, we consider the function

$$G_\alpha\colon \mathbb{Z}^m \times M \to \mathbb{Z}^m \times M, \quad G_\alpha(t,x) = \big(t + 1_\alpha, F(t,x)\big),\ \ \forall (t,x) \in \mathbb{Z}^m \times M.$$

Similar to the proof of Proposition 5, it is shown that the relations (31) are true, for any $(t,x) \in \mathbb{Z}^m \times M$ if and only if $G_\alpha \circ G_\beta = G_\beta \circ G_\alpha$.

From Lemma 3, we deduce that, for any $\alpha \in \{1, 2, \ldots, m\}$, the function G_α is bijective.

Let $(t_0, x_0) \in \mathbb{Z}^m \times M$. According the Theorem 2, *vi*), there exists a unique function $y(\cdot) = (s(\cdot), x(\cdot))\colon \mathbb{Z}^m \to \mathbb{Z}^m \times M$, which, for any $t \in \mathbb{Z}^m$ verifies the relations

$$y(t + 1_\alpha) = G_\alpha(y(t)), \quad \forall \alpha \in \{1, 2, \ldots, m\}, \tag{32}$$

and $y(t_0) = (t_0, x_0)$, which are equivalent to

$$\begin{cases} x(t + 1_\alpha) = F_\alpha(s(t), x(t)) \\ s(t + 1_\alpha) = s(t) + 1_\alpha \end{cases}, \quad \forall \alpha \in \{1, 2, \ldots, m\} \tag{33}$$

and $s(t_0) = t_0$, $x(t_0) = x_0$.

From Lemma 2, we obtain $s(t) = t$, $\forall t \in \mathbb{Z}^m$. Replacing in the first relation of (33), it follows that the function $x\colon \mathbb{Z}^m \to M$ verifies the relations (30), $\forall t \in \mathbb{Z}^m$.

Uniqueness of $x(\cdot)$: let $\tilde{x}\colon \mathbb{Z}^m \to M$ be a function which verifies the relations (30), $\forall t \in \mathbb{Z}^m$, and the condition $\tilde{x}(t_0) = x_0$. Easily finds that the function

$$\tilde{y}\colon \mathbb{Z}^m \to \mathbb{Z}^m \times M, \quad \tilde{y}(t) = (t, \tilde{x}(t)), \quad \forall t \in \mathbb{Z}^m,$$

verifies the relations (32), $\forall t \in \mathbb{Z}^m$, and the condition $\tilde{y}(t_0) = (t_0, x_0)$.

From the uniqueness property of solutions of the recurrence (32) (according Theorem 2, *vi*)) it follows that the functions y and $\tilde{y}$ coincide; hence $\big(t, x(t)\big) = \big(t, \tilde{x}(t)\big)$, $\forall t \in \mathbb{Z}^m$; we obtain $x(t) = \tilde{x}(t)$, $\forall t \in \mathbb{Z}^m$.

iv) $\Longrightarrow$ *iii*): For each $\alpha \in \{1, 2, \ldots, m\}$, we consider the function G_α defined as in the proof of the implication *i*) $\Longrightarrow$ *iv*).

Let $t_0 \in \mathbb{Z}^m$ and $(s_0, x_0) \in \mathbb{Z}^m \times M$. We shall show that there exists a unique function $y(\cdot) = (s(\cdot), x(\cdot))\colon \mathbb{Z}^m \to \mathbb{Z}^m \times M$, which, for any $t \in \mathbb{Z}^m$, verifies the relations (32), and $y(t_0) = (s_0, x_0)$.

There exists a unique function $\tilde{x}\colon \mathbb{Z}^m \to M$ which, for any $t \in \mathbb{Z}^m$ verifies the relations (30), and also the condition $\tilde{x}(s_0) = x_0$.

Let

$$s\colon \mathbb{Z}^m \to \mathbb{Z}^m, \quad x\colon \mathbb{Z}^m \to M,$$

$$s(t) = t - t_0 + s_0, \quad x(t) = \tilde{x}(t - t_0 + s_0), \quad \forall t \in \mathbb{Z}^m.$$

Easily finds that the function $y(\cdot) = (s(\cdot), x(\cdot))\colon \mathbb{Z}^m \to \mathbb{Z}^m \times M$ verifies, for any $t \in \mathbb{Z}^m$, the recurrence (32) and $y(t_0) = (s_0, x_0)$.

The uniqueness of $y(\cdot)$: Let $\tilde{y}(\cdot) = (\sigma(\cdot), z(\cdot))\colon \mathbb{Z}^m \to \mathbb{Z}^m \times M$ which verifies, for any $t \in \mathbb{Z}^m$, the recurrence (32) and the condition $\tilde{y}(t_0) = (s_0, x_0)$. Hence, for any $t \in \mathbb{Z}^m$, the functions $\sigma(\cdot)$, $z(\cdot)$ verify the recurrence (33) and $\sigma(t_0) = s_0$, $z(t_0) = x_0$. From Lemma 2, it follows $\sigma(t) = t - t_0 + s_0 = s(t)$, $\forall t \in \mathbb{Z}^m$.

One observes immediately that the function

$$\tilde{z}\colon \mathbb{Z}^m \to M, \quad \tilde{z}(t) = z(t + t_0 - s_0), \quad \forall t \in \mathbb{Z}^m,$$

verifies the recurrence (30), $\forall t \in \mathbb{Z}^m$, and $\tilde{z}(s_0) = x_0$. It follows that the functions $\tilde{x}$ and $\tilde{z}$ coincide. Hence, $\forall t \in \mathbb{Z}^m$, we have $\tilde{z}(t - t_0 + s_0) = \tilde{x}(t - t_0 + s_0)$, i.e., $z(t) = x(t)$.

Since $\sigma(\cdot) = s(\cdot)$ and $z(\cdot) = x(\cdot)$, we have $\tilde{y}(\cdot) = y(\cdot)$.

Hence, for the recurrence (32) we can apply the Theorem 2, implication *vi*) $\Longrightarrow$ *iv*).

Let $t_0, t_1 \in \mathbb{Z}^m$, with $t_1 \leq t_0$, and $x_0 \in M$.

There exists a unique function $\tilde{x}\colon \mathbb{Z}^m \to M$, such that $\tilde{x}(t_0) = x_0$ and the relations (30) are true, $\forall t \in \mathbb{Z}^m$. It is sufficient to select x as being the restriction of $\tilde{x}$ to $\{t \in \mathbb{Z}^m \,|\, t \geq t_1\}$.

Uniqueness of the function $x(\cdot)$: Let $z\colon \{t \in \mathbb{Z}^m \,|\, t \geq t_1\} \to M$, which, for any $t \geq t_1$ verifies the recurrence (30), and $z(t_0) = x_0$.

Let $y, \tilde{y}\colon \{t \in \mathbb{Z}^m \,|\, t \geq t_1\} \to \mathbb{Z}^m \times M$,

$$y(t) = (t, x(t)), \quad \tilde{y}(t) = (t, z(t)), \quad \forall t \geq t_1.$$

One observes immediately that y and $\tilde{y}$ verify the recurrence (32), $\forall t \geq t_1$; we have also $y(t_0) = \tilde{y}(t_0) = (t_0, x_0)$. From Theorem 2, it follows that the functions y and $\tilde{y}$ coincide. We obtain $(t, x(t)) = (t, z(t))$, $\forall t \geq t_1$; hence $x(t) = z(t)$, $\forall t \geq t_1$.

iii) $\Longrightarrow$ *ii)* is an obvious implication. $\square$

4 Example

Let M be a nonvoid set, $(N, \cdot, e)$ be a monoid and let $\varphi\colon N \times M \to M$ be an action of the monoid N on the set M, i.e.,

$$\varphi(ab, x) = \varphi\big(a, (b, x)\big), \;\; \varphi(e, x) = x, \quad \forall a, b \in N, \forall x \in M. \tag{34}$$

For each $a \in N$, $x \in M$, we denote $\varphi(a, x) = ax$ (not to be confused with the monoid operation N). The relations (34) become

$$(ab)x = a(bx), \;\; ex = x, \quad \forall a, b \in N, \forall x \in M.$$

We consider $a_1, a_2, \ldots, a_m \in N$, such that $a_\alpha a_\beta = a_\beta a_\alpha$, $\forall \alpha, \beta \in \{1, 2, \ldots, m\}$.

For each pair $(t_0, x_0) \in \mathbb{Z}^m \times M$, the recurrence

$$x(t + 1_\alpha) = a_\alpha x(t), \quad \forall \alpha \in \{1, 2, \ldots, m\}, \tag{35}$$

with the initial condition $x(t_0) = x_0$, has unique solution

$$x\colon \{t \in \mathbb{Z}^m \,|\, t \geq t_0\} \to M,$$

$$x(t) = a_1^{(t^1 - t_0^1)} a_2^{(t^2 - t_0^2)} \cdot \ldots \cdot a_m^{(t^m - t_0^m)} x_0. \tag{36}$$

This can be obtained by applying Theorem 1 for the functions $G_\alpha\colon M \to M$, $G_\alpha(x) = a_\alpha x$, $\forall x \in M$. We have $G_\alpha \circ G_\beta(x) = G_\beta \circ G_\alpha(x) = a_\alpha a_\beta x$. One observes that, for any $t \in \mathbb{N}^m$,

$$G_1^{(t^1)} \circ G_2^{(t^2)} \circ \ldots \circ G_m^{(t^m)}(x) = a_1^{t^1} a_2^{t^2} \cdot \ldots \cdot a_m^{t^m} x. \tag{37}$$

Suppose that for any $\alpha \in \{1, 2, \ldots, m\}$, a_α is invertible; then G_α is bijective, with the inverse $G_\alpha^{-1}(x) = a_\alpha^{-1}x$. We find that the formula (37) is true for any $t \in \mathbb{Z}^m$. According the Remark 1, there exists a unique function $\tilde{x}\colon \mathbb{Z}^m \to M$, solution of the recurrence (35), with $\tilde{x}(t_0) = x_0$; the function $\tilde{x}(\cdot)$ is a unique extension of $x(\cdot)$ and it is defined by the formula (36), but for each $t \in \mathbb{Z}^m$.

Particular cases:

1) Let $(M, +)$ be a commutative monoid; we consider the action of M on himself

$$\varphi\colon M \times M \to M, \quad \varphi(a, x) = a + x, \quad \forall a, x \in M.$$

In this case the recurrence (35) becomes

$$x(t + 1_\alpha) = a_\alpha + x(t), \quad \forall \alpha \in \{1, 2, \ldots, m\},$$

and the formula (36) can be written

$$x(t) = (t^1 - t_0^1)a_1 + (t^2 - t_0^2)a_2 + \ldots + (t^m - t_0^m)a_m + x_0.$$

2) Let K be a field.

We consider $\big(N, \cdot, e\big) = \big(\mathcal{M}_n(K), \cdot, I_n\big)$, $M = K^n = \mathcal{M}_{n,1}(K)$ and the action

$$\varphi\colon \mathcal{M}_n(K) \times K^n \to K^n, \quad \varphi(A, x) = Ax, \quad \forall A \in \mathcal{M}_n(K),\ \forall x \in K^n.$$

Let $A_1, A_2, \ldots, A_m \in \mathcal{M}_n(K)$, such that $A_\alpha A_\beta = A_\beta A_\alpha, \forall \alpha, \beta \in \{1, 2, \ldots, m\}$.

It follows that, for each pair $(t_0, x_0) \in \mathbb{Z}^m \times K^n$, the recurrence

$$x(t + 1_\alpha) = A_\alpha x(t), \quad \forall \alpha \in \{1, 2, \ldots, m\}, \tag{38}$$

with the initial condition $x(t_0) = x_0$, has unique solution

$$x\colon \big\{t \in \mathbb{Z}^m \,\big|\, t \geq t_0\big\} \to K^n,$$

$$x(t) = A_1^{(t^1 - t_0^1)} A_2^{(t^2 - t_0^2)} \cdot \ldots \cdot A_m^{(t^m - t_0^m)} x_0. \tag{39}$$

If, for each $\alpha \in \{1, 2, \ldots, m\}$, the matrix A_α is invertible, then there exists a unique function $\tilde{x}\colon \mathbb{Z}^m \to K^n$, solution of the recurrence (38), with $\tilde{x}(t_0) = x_0$; the function $\tilde{x}(\cdot)$ is a unique extension of $x(\cdot)$ and it is defined by the formula (39), but for each $t \in \mathbb{Z}^m$.

Acknowledgments

The work has been funded by the Sectoral Operational Programme Human Resources Development 2007-2013 of the Ministry of European Funds through the Financial Agreement POSDRU/159/1.5/S/132395.

Partially supported by University Politehnica of Bucharest and by Academy of Romanian Scientists. Special thanks goes to Prof. Dr. Ionel Ţevy, who was willing to participate in our discussions about multivariate sequences and to suggest the title "multiple recurrences".

References

[1] M. Ahmadi, Design of 2-Dimensional recursive digital filters, Control and Dynamics System 78 (1996) 131-181.

[2] M. Bousquet-Mélou, M. Petkovšek, Linear recurrences with constant coefficients: the multivariate case, Discrete Mathematics 225 (1) (2000) 51-75.

[3] D. Dudgeon, R. Mersereau, Multidimensional Digital Signal Processing, Prentice-Hall Signal Processing Series, 1983.

[4] S. Elaydi, An Introduction to Difference Equations, Springer, 2005.

[5] E. Fornasini, G. Marchesini, Doubly-indexed dynamical systems: state-space models and structural models, Math. Systems Theory 12 (1978) 59-72.

[6] E. Fornasini, G. Marchesini, Stability analysis of 2-D systems, IEEE Trans. Circuits and Systems 27 (12) (1980), 1210-1217.

[7] C. Ghiu, R. Tuligă, C. Udrişte, I. Ţevy, Discrete multitime recurrences and discrete minimal submanifolds, Balkan J. Geom. Appl. 20 (2015), will appear.

[8] C. Ghiu, R. Tuligă, C. Udrişte, I. Ţevy, Discrete multitime recurrences and their application in economics, The VIII-th International Conference "Differential Geometry and Dynamical Systems" (DGDS-2014) September 1 - 4, 2014, Mangalia, Romania.

[9] H. Hauser, C. Koutschan, Multivariate linear recurrences and power series division, Discrete Mathematics 312 (2012) 3553-3560.

[10] T. S. Huang, Stability of two-dimensional recursive filters, IEEE Transactions on Audio and Electroacoustics 20 (2) (1972) 158-163.

[11] J. S. Lim, Two-Dimensional Signal and Image Processing, Prentice-Hall Inc., Englewood Cliffs, NJ, 1990.

[12] N. Mastorakis, I. Gonos, M. Swamy, Design of two-dimensional recursive filters using genetic algorithms, IEEE Transactions on Circuits and Systems I Fundamental Theory and Applications 50 (5) (2003) 634-639.

[13] A. Mazinani, M. Ahmadi, M. Shridhar and R. S. Lashkari, A novel approach to the design of 2-D recursive digital filters, Journal of the Franklin Institute, Pergamon Press Ltd 329 (1) (1992) 127-133.

[14] V. Prepeliţă, Multiple (n, m)-hybrid Laplace Transformation and applications to multidimensional hybrid systems. Part I, U.P.B. Sci. Bull., Series A 72 (2) (2010) 105-120.

[15] C. Udrişte, Multitime maximum principle for curvilinear integral cost, Balkan J. Geom. Appl. 16 (1) (2011) 128-149.

[16] C. Udrişte, A. Bejenaru, Multitime optimal control with area integral costs on boundary, Balkan J. Geom. Appl. 16 (2) (2011) 138-154.

[17] C. Udrişte, I. Ţevy, Multitime dynamic programming for multiple integral actions, Journal of Global Optimization 51 (2) (2011) 345-360.

[18] C. Udrişte, Minimal submanifolds and harmonic maps through multitime maximum principle, Balkan J. Geom. Appl. 18 (2) (2013) 69-82.

[19] C. Udrişte, S. Dinu, I. Ţevy, Multitime optimal control for linear PDEs with curvilinear cost functional, Balkan J. Geom. Appl. 18 (1) (2013) 87-100.

[20] Jr. Webber, L. Charles, N. Marvan (Eds.), Recurrence Quantification Analysis, Springer, 2015.